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APPLICATIONS OF A THEOREM REGARDING CIRCULANTS.

By HARRY S. VANDIVER, Bala, Pa.

A circulant of the nth order is a simple determinant of the following type:

$$\begin{vmatrix} a_1 & \omega_n a_2 & \omega_n^2 a_3 \ldots\ldots & \omega_n^{n-1} a_n \\ \omega_n^{n-1} a_n & a_1 & \omega_n a_2 \ldots\ldots & \omega_n^{n-2} a_{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ \omega_n a_2 & \omega_n^2 a_3 & \omega_n^3 a_4 \ldots\ldots & a_1 \end{vmatrix}$$

where ω_n is a special nth root of unity. This is the product of n linear factors, as follows:

$$\begin{array}{l} a_1 + \omega_n a_2 + \omega_n^2 a_3 + \omega_n^3 a_4 \;\ldots\ldots\; \omega_n^{n-1} a_n \\ a_1 + \omega_n^2 a_2 + \omega_n^4 a_3 + \omega_n^6 a_4 \;\ldots\ldots\; \omega_n^{n-2} a_n \\ a_1 + \omega_n^3 a_2 + \omega_n^6 a_3 + \omega_n^9 a_4 \;\ldots\ldots\; \omega_n^{n-3} a_n \\ \cdot \quad\; \cdot \quad\; \cdot \quad\; \cdot \quad\; \cdot \quad\; \cdot \\ \cdot \quad\; \cdot \quad\; \cdot \quad\; \cdot \quad\; \cdot \quad\; \cdot \\ a_1 + a_2 + a_3 \;\ldots\ldots\ldots\ldots\; a_n \end{array}$$

A theorem regarding such functions is:

The product of any number of circulants each of the order n can be expressed as a circulant of the nth order.

For simplicity, we will prove this for continuants of the third order; the procedure is precisely the same when $n>3$. We have:

$$\begin{vmatrix} a_1 & \omega_3 a_2 & \omega_3^2 a_3 \\ \omega_3^2 a_3 & a_1 & \omega_3 a_2 \\ \omega_3 b_2 & \omega_3^2 a_3 & a_1 \end{vmatrix} \times \begin{vmatrix} b_1 & \omega_3 b_2 & \omega_3^2 b_3 \\ \omega_3^2 b_3 & b_1 & \omega_3 b_2 \\ \omega_3 b_2 & \omega_3^2 b_3 & b_1 \end{vmatrix}$$

This equals the product of the six factors:

$$\begin{array}{l} (a_1 + a_2 + a_3)(b_1 + b_2 + b_3) \\ (a_1 + \omega_3 a_2 + \omega_3^2 a_2)(b_1 + \omega_3 b_2 + \omega_3^2 b_3) \\ (a_1 + \omega_3^2 a_2 + \omega_3 a_3)(b_1 + \omega_3^2 b_2 + \omega_3 b_3) \end{array}$$

Taken two and two, these give:

$$\begin{array}{l} A_1 + A_2 + A_3 \\ A_1 + \omega_3 A_2 + w_3^2 A_3 \\ A_1 + \omega_3^2 A_2 + \omega_3 A_3. \end{array}$$

Whence the theorem.

If it is true for two circulants it is true for any number.

Corollary. Any integral power of a circulant of the nth order is a circulant of the same order.

Some of the applications of this theorem and corollary to Diophantine analysis follow:

$$\begin{vmatrix} x & iy \\ iy & x \end{vmatrix} \times \begin{vmatrix} a & ib \\ ib & a \end{vmatrix} = \begin{vmatrix} c & id \\ id & c \end{vmatrix}$$

or $(x^2+y\ )(a^2+b^2)=c^2+d^2$, a well known theorem.

If the relation $x^2+bxy+cy^2=1$ *possesses one solution in integers it possesses an infinite number.*

The function $x^2+bxy+cy^2$ can be expressed as a circulant as follows:

$$\begin{vmatrix} \dfrac{2x+by}{2} & \dfrac{\sqrt{(b^2-ac)}}{2}y \\ \dfrac{\sqrt{(b^2-4ac)}}{2}y & \dfrac{2x+by}{2} \end{vmatrix}$$

Suppose the equation possesses one solution, say $x=m$, $y=n$.

Then $m^2+bmn+cn^2=1$; then we can put

$$(m^2+bmn+cn^2)^k=x^2+bxy+cy^2.$$

Putting $k=2$, and decomposing both circulants into their irrational factors, there is obtained:

$$\frac{2x+by-y\sqrt{(b^2-4c)}}{2}=\left[\frac{2m+bn-n\sqrt{(b^2-4c)}}{2}\right]^2$$

and

$$\frac{2x+by+y\sqrt{(b^2-4c)}}{2}=\left[\frac{2m+bn+n\sqrt{(b^2-4c)}}{2}\right]^2$$

Taking the first relation and expanding

$$4x+2by-2y\sqrt{(b^2-4c)}=(2m+bn)^2-2n(2m-bn)\sqrt{(b^2-4c)}+n^2(b^2-4c).$$

Equating rational and irrational parts

$$4x+2by=(2m+bn)^2+n^2(b^2-4c) \text{ and } y=n(2m+bn).$$

By substitution we find $x=m^2-cn^2$.

Proceeding in the same manner when $k>2$, we find integral values for x and y. Hence since k is unlimited the number of solutions is unlimited. Hence the theorem.

To show that, for every value of n, the relation

$$x^3+ay^3+a^2z^3-3axyz=v^n$$

has (when a is an integer) an infinite number of integral solutions in x, y, and z.

The function $x^3+ay^3+a^2z^3-3axyz$ expressed as a circulant is:

$$\begin{vmatrix} x & y\sqrt[3]{a} & z\sqrt[3]{a^2} \\ z\sqrt[3]{a^2} & x & y\sqrt[3]{a} \\ y\sqrt[3]{a} & z\sqrt[3]{a^2} & x \end{vmatrix}$$

We then can assume $(x_1^3+ay_1^3+a^2z_1^3-3ax_1y_1z_1)^n=x^3+ay^3+a^2z^3-3axyz$.

Put $n=2$. Then

$$(x_1+y_1\sqrt[3]{a}+z_1\sqrt[3]{a^2})^2=x+y\sqrt[3]{a}+z\sqrt[3]{a^3}.$$

From which, by expanding and equating irrationals,

$$x=x_1^2+2ay_1z_1$$
$$y=z_1^2a+2x_1y_1$$
$$z=y_1^2+2x_1y_1.$$

The number of values that can be assigned to x_1, y_1, z_1 is infinite; hence the theorem is proved for $n=2$. When $n>2$ it may be proved by proceeding in the same manner.

In Article 9, entitled, "Of the manner of finding Algebraic Functions of all degrees, which when multiplied together may always produce similar functions" in the Additions to Euler's Algebra, Lagrange gives developments which may be considered as applications of the circulant theorem.

Bala, Pa., July 13, 1901.

THE BETWEENNESS ASSUMPTIONS.

By DR. GEORGE BRUCE HALSTED.

In his "Vorlesung ueber Euklidische Geometrie" (Wintersemester 1898 99), Hilbert gave the most remarkable set of axioms ever created for the founding of geometry. These were published in his Festschrift, 1899, treated of in my "Supplementary Report" to the A. A. A. S., (*Science*, Nov. 8, 1901). They have been given in English, though in part erroneously, in D. E. Smith's "The Teaching of Elementary Mathematics," (Macmillan).